

# Unified Quantification of Nonclassicality and Non-Gaussianity: An Entropic Approach

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Nonclassical states of quantized light, except for the Gaussian states, also possess non-Gaussian phase-space distributions. Despite several attempts, a unified description of nonclassicality ( $NC$ ) and non-Gaussianity ( $NG$ ) of quantum states of light has not been developed as-of-yet. Here, we propose an experimentally verifiable scheme for quantification of  $NC$ , in terms of Wehrl entropy, that further leads to the simultaneous quantification of  $NC$  and  $NG$ . While requiring no optimization, present work recovers earlier results qualitatively as well as explores several new possibilities on the conjugation of nonclassical and non-Gaussian character of quantum states. Moreover, current formalism, due to its possible extension to the finite-dimensional systems, bridges the gap between discrete and continuous variable systems. Our work, thus, becomes crucial in describing  $NC$  of quantum processes including open quantum systems as well as understanding the role of  $NC$  and  $NG$  as resources in several information theoretic tasks processing such as entanglement distillation, quantum network, quantum computation *etc.*

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Quantum states of light, unlike the classical electromagnetic wave, show many intriguing and fascinating phenomena like photon antibunching, sub-Poissonian distribution, oscillatory number distribution *etc.* [1]. Despite obtained through quantization of the electromagnetic field, some of them possess nonclassical phase-space distributions leading to further classification of the states as classical and nonclassical. For any quantum state of light  $\rho$ , if the phase-space Glauber-Sudarshan  $P$  distribution [2],  $\rho = \int \frac{d^2\alpha}{\pi} P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|$ , behaves like a classical probability distribution, i.e., positive semidefinite or singular no more than a delta function,  $\rho$  is said to be classical, otherwise nonclassical, where  $|\alpha\rangle$  stands for coherent state and  $\alpha$  is any complex number. Such nonclassical states could be generated by several nonclassicality ( $NC$ )-inducing operations such as photon excitation [3], quadrature squeezing [4], Kerr squeezing [5] *etc.* applied on a classical state [6],  $|\beta\rangle$ . Alongwith nonclassical  $P$  distribution, some of the quantum states have non-Gaussian Wigner distribution  $W(z, z^*) = 2e^{2|z|^2} \int \frac{d^2\alpha}{\pi} \langle -\alpha | \rho | \alpha \rangle e^{2(z\alpha^* - z^*\alpha)}$  [7] which further introduces a notion of non-Gaussianity ( $NG$ ), where  $z$  and  $\alpha$  being complex numbers. For instance,  $|\beta\rangle$  has a Gaussian  $W(z, z^*)$ ; however, photon excitation and Kerr squeezing applied on  $|\beta\rangle$ , leads to non-Gaussian  $W(z, z^*)$  while quadrature squeezing leaves the Gaussian character unchanged. Since  $W(z, z^*)$  can be obtained from  $P(\alpha, \alpha^*)$  by the Gaussian convolution,  $W(z, z^*) = 2 \int \frac{d^2\alpha}{\pi} P(\alpha, \alpha^*) e^{-2|z-\alpha|^2}$  [1], we ask, how much nonclassical as well as non-Gaussian does a classical state become under the action of such  $NC$ -inducing operations? In other words, whether there exists any unified description of  $NC$  and  $NG$  of the quantum states of light?

Several attempts have been made to quantify [9–13] as well detect [14–17] the  $NC$  of single mode quantum

state of light in quantum Hilbert space [9], phase-space [10, 14] alongside exploiting the BS convertibility of single mode  $NC$  into output entanglement [13]. Moreover, there are both Hilbert space and phase-space based descriptions for quantification [18–20] and detection [21, 22] of  $NG$  as well. Recently, Park *et al.* [23] have proposed a unique protocol to detect the  $NC$  and  $NG$  of any quantum state of light in terms of the nofactorizability of the corresponding Wigner distribution; however, a unified quantitative description of the same has not been developed as-of-yet.

Here, we propose a concurrent description of  $NC$  and  $NG$  for a single mode quantum optical state in terms of the associated Husimi-Kano  $Q$  distribution [1]  $Q(\beta, \beta^*) (= \langle \beta | \rho | \beta \rangle)$ , the ‘classical’ like distribution [24]. We describe the  $NC$  of any quantum optical state  $\rho_{nc}$  as the difference between the Wehrl entropy of  $\rho_{nc}$ ,  $H_w(\rho_{nc}) = - \int \frac{d^2\beta}{\pi} Q_{\rho_{nc}}(\beta, \beta^*) \log Q_{\rho_{nc}}(\beta, \beta^*)$ , and that of its nearest classical state, which further leads to the unified quantization of  $NC$  and  $NG$ . We analyze both pure and mixed states to test the validity of our formalism. In case of pure state, we show that, for a photon number state and a squeezed coherent state,  $NC$  depends on number of photon excitation and squeezing strength monotonically. For Schrodinger kittens [25], we detect the odd superposition is more nonclassical than the even superposition; however, with increase in the coherent amplitude, as we approach macroscopic cat states, for both the states,  $NC$  becomes equal and saturates while the  $NG$  increases monotonically. This interplay between  $NC$  and  $NG$  leads to an interesting situation, in the context of conjugation of  $NC$  and  $NG$ , where the quantum states, in particular, coherent superposition states, could have  $NC$  more as well as less than its  $NG$ . For mixed states, our formalism theoretically predicts a new paradigm where the nearest Gaussian state could

be more nonclassical than the state itself. For photon excited thermal noise, present formulation encodes the  $NC$  that is monotonic on number of photon excitation while being completely independent on the initial noise strength, whereas, in the case of squeezed state  $NC$  still depends on the initial noise. It is noteworthy that, for squeezed thermal state, we recover the reference thermal state same as the initial chaotic field, as considered in [10] while detecting  $NC$  for all non-vanishing squeezing strengths unlike reported earlier [13]. Although predicted on theoretical ground, we leave the query of nonclassical mixed state having  $NC$  less than its nearest Gaussian state, open.

*NC of Quantum States* : We define  $NC$  of any quantum state  $\rho_{nc}$  as the difference between its Wehrl entropy and that of the nearest classical state as,

$$NC(\rho_{nc}) = \inf_{\rho_{cl}} |H_w(\rho_{nc}) - H_w(\rho_{cl})|, \quad (1)$$

where the infimum has been considered over the set of all classical states. We choose the optimized classical reference state in Eq. (1) as follows.

Since the coherent states  $|\beta\rangle$  are the only classical pure states [6], in the case of pure states, we choose coherent states ( $|\beta\rangle$ ) as the set of classical reference state. Besides, in any optical experiment with decoherence, quantum states, by loosing all the quantum correlations, end up at the thermal state  $\rho(\bar{n})$  which is a classical mixed state. Consequently, for mixed states, we consider thermal states as the set of classical reference states. Furthermore, since both  $|\beta\rangle$  and  $\rho(\bar{n})$  are explicitly Gaussian in nature, we exploit the property of Gaussian distributions to bypass the optimization. For any bi-variate Gaussian distribution,  $G(\vec{x}) = \sqrt{\det[\sigma^{-1}]} e^{-(\vec{x}-\vec{\mu})^T \sigma^{-1} (\vec{x}-\vec{\mu})}$ , its entropy,  $E(G(\vec{x})) = -\int \frac{d^2\vec{x}}{\pi} G(\vec{x}) \log G(\vec{x})$  is given by  $E(G(\vec{x})) = 1 + \log \sqrt{\det[\sigma]}$ ; where  $\sigma$  and  $\vec{\mu}$  are the variance matrix and the displacement vector for the distribution respectively. Thus, the entropy of any Gaussian distribution is dependent only on its variance matrix, not the displacement.

Any coherent state  $|\beta\rangle$  is a displaced vacuum state  $[|\beta\rangle = D(\beta)|0\rangle]$  which leads  $H_w(|\beta\rangle) = H_w(|0\rangle) = 1.0$ . Moreover, the minimum entropy of any classical distribution is unity [26]. As a consequence,  $NC$  (Eq. 1) for all pure states reduces to,

$$NC(|\psi\rangle_{nc}) = H_w(|\psi\rangle_{nc}) - 1.0 \quad (2)$$

As Evident from Eq. (2), any classical pure state  $|\beta\rangle$  has a vanishing  $NC$ . On the other hand, Wigner distribution for any Gaussian nonclassical state  $\rho^G$  has the form,

$$W^G(z, z^*) = \frac{\exp\left(-\frac{\mu(z-z_0)^2 + \mu^*(z^*-z_0^*)^2 + \tau|z-z_0|^2}{\tau^2 - 4|\mu|^2}\right)}{\sqrt{\tau^2 - 4|\mu|^2}}, \quad (3)$$

with  $\det[\sigma] = \tau^2 - 4|\mu|^2$ , where  $z_0 = \langle a \rangle_{\rho^G}$ ,  $\tau = \langle a^\dagger a \rangle_{\rho^G} - |z_0|^2 + \frac{1}{2}$ ,  $2\mu = \langle a^{\dagger 2} \rangle_{\rho^G} - z_0^{*2}$  and  $\sigma$  stands for the variance

matrix. For a thermal state  $\rho_{th}(\tilde{n})$ , a special case of  $\rho^G$  ( $\mu = 0, z_0 = 0$ ), Eq. (3) boils down to

$$W_{th}^{\tilde{n}}(z, z^*) = \frac{1}{\tilde{\tau}} \exp\left(-\frac{|z|^2}{\tilde{\tau}}\right), \quad (4)$$

with  $\det[\sigma] = \tilde{\tau}$ , where  $\tilde{\tau} = \langle a^\dagger a \rangle_{\rho_{th}} + \frac{1}{2} = \tilde{n} + \frac{1}{2}$ .

Now we need to set the thermal reference state that is nearest to  $\rho_{nc}$ . Since, we describe  $NC$  in terms of entropy, finding the nearest  $\rho_{th}(\tilde{n})$  is equivalent to figuring out the very  $\rho_{th}(\tilde{n})$  that has maximal entropy w.r.t.  $\rho_{nc}$ . The Gaussian state  $\rho_{nc}^G$  that is nearest to  $\rho_{nc}$  has the first and the second moment same as that of the  $\rho_{nc}$  itself [27]. Owing to the fact that a thermal state has a Gaussian  $W(z, z^*)$  (Eq. 4), we can choose the nearest thermal state  $\rho_{th}(\tilde{n})$  to be the very state which has maximal entropy compared to  $\rho_{nc}$ , e.g., entropy equal to that of  $\rho_{nc}^G$  at the Wigner level. Considering the expression for entropy of a Gaussian distribution, we obtain the desired state by equating the value of determinant of  $\sigma$  of the respective  $W(z, z^*)$  for  $\rho_{th}(\tilde{n})$  (Eq. 4) and  $\rho_{nc}^G$  (Eq. 3), i.e., by setting  $\tilde{\tau} = \sqrt{\tau^2 - 4|\mu|^2}$ , where  $\tau = \langle a^\dagger a \rangle_{\rho_{nc}} - |z_0|^2 + \frac{1}{2}$ ,  $2\mu = \langle a^{\dagger 2} \rangle_{\rho_{nc}} - z_0^{*2}$  and  $z_0 = \langle a \rangle_{\rho_{nc}}$ .

Furthermore, using the relation between  $W(z, z^*)$  and  $Q(\beta, \beta^*)$ ,  $Q_\rho(\alpha, \alpha') = 2 \int \frac{d^2\beta}{\pi} W_\rho(\beta, \beta^*) e^{-2|\alpha-\beta|^2}$ , it is easy to show that  $\rho^G$ , given by the  $W^G(z, z^*)$  (Eq. 3), has Wehrl entropy of the form  $H_w(\rho^G) = 1 - \log 2 + \frac{1}{2} \log [4(\tau^2 - 4|\mu|^2) + 4\tau + 1]$  while that for  $\rho_{th}(\tilde{n})$ , described by  $W_{th}^{\tilde{n}}(z, z^*)$  (Eq. 4), becomes  $H_w(\rho_{th}(\tilde{n})) = 1 - \log 2 + \log [2\tilde{\tau} + 1]$ . For the sake of convenience and clarity let's denote the nearest thermal reference state as  $\rho_{th}^{nc}(\tilde{n})$ . Thus, we obtain  $NC$  (Eq. 1) of mixed states in the form,

$$NC(\rho_{nc}) = |H_w(\rho_{nc}) - H_w(\rho_{th}^{nc})| \\ = |H_w(\rho_{nc}) - 1 + \log 2 - \log [2\tilde{\tau} + 1]| \quad (5)$$

*Simultaneous Description of NC and NG* : Now we address the issue of simultaneous quantification of  $NC$  and  $NG$ . We rewrite the definition of  $NC$  for pure states (Eq. 2) as,

$$NC(|\psi\rangle_{nc}) = H_w(|\psi\rangle_{nc}) - H_w(\rho_{nc}^G) + H_w(\rho_{nc}^G) - 1.0 \\ = (H_w(\rho_{nc}^G) - 1.0) - (H_w(\rho_{nc}^G) - H_w(|\psi\rangle_{nc})) \\ = \Delta(\rho_{nc}^G) - NG(|\psi\rangle_{nc}) \quad (6)$$

where  $\rho_{nc}^G$  is the nearest Gaussian state w.r.t  $|\psi\rangle_{nc}$  [27] and  $NG(|\psi\rangle_{nc})$  is the  $NG$  of  $|\psi\rangle_{nc}$  [20]. Similarly we rewrite Eq. (5) as  $NC(\rho_{nc}) = |(H_w(\rho_{nc}^G) - H_w(\rho_{th}^{nc}) - (H_w(\rho_{nc}^G) - H_w(\rho_{th}^{nc})))|$  that yields,

$$NC(\rho_{nc}) \geq |H_w(\rho_{nc}^G) - H_w(\rho_{th}^{nc})| - NG(\rho_{nc}) \\ \geq |\Delta(\rho_{nc}^G) - NG(\rho_{nc})| \quad (\because |a-b| \geq ||a| - |b||), \quad (7)$$

where  $NG(\rho_{nc}) = H_w(\rho_{nc}^G) - H_w(\rho_{nc})$  is the  $NG$  of  $\rho_{nc}$  [20]. In the definition of  $NC$  (Eq. 1) of any quantum

optical state  $\rho_{\text{nc}}$  we look for the minimal entropy that the state has in addition to its classical entropy which can be attributed solely to its nonclassical character. As a consequence, any non-Gaussian classical state must have vanishing  $NC$  while its  $NG$  still being non-zero. Thus, we consider the equality in Eq. (7).

Moreover, depending upon whether  $\rho_{\text{nc}}$  is the mixed,  $\Delta(\rho_{\text{nc}}^G)$  attains further interpretation. For pure states,  $\Delta(\rho_{\text{nc}}^G) = H_w(\rho_{\text{nc}}^G) - 1$  while for mixed states,  $\Delta(\rho_{\text{nc}}^G) = H_w(\rho_{\text{nc}}^G) - H_w(\rho_{\text{cl}}) = NC(\rho_{\text{nc}}^G)$ . As a consequence, we obtain the simultaneous quantification of  $NC$  and  $NG$  in the form

$$\begin{aligned} NC(|\psi\rangle_{\text{nc}}) &= \Delta(\rho_{\text{nc}}^G) - NG(|\psi\rangle_{\text{nc}}) \\ NC(\rho_{\text{nc}}) &= |\Delta(\rho_{\text{nc}}^G) - NG(\rho_{\text{nc}})|. \end{aligned} \quad (8)$$

As evident from Eq. (8) we explore new possibilities such as (i) any quantum state could have  $NC$  more as well as less than its  $NG$ , (ii) for mixed states,  $\rho_{\text{nc}}^G$  could be more nonclassical than  $\rho_{\text{nc}}$ . Next, we illustrate our formalism for certain pure and mixed quantum states of light.

**Pure States :** We consider photon number state and squeezed coherent state as examples of non-Gaussian and Gaussian pure states respectively. We also study even and the odd superposition of coherent states as examples of general superposition state.

Photon number state  $|m\rangle$  is obtained by applying photon excitation ( $\frac{a^{\dagger m}}{\sqrt{m!}}$ ) on the vacuum which is a special case of coherent state  $|\alpha\rangle$  ( $\alpha = 0$ ). For  $|m\rangle$ , we have  $H_w(|m\rangle) = 1 + m + \log m! - m\Psi(m+1)$  [20];  $\Psi(m+1) = \sum_{k=1}^m \frac{1}{k} - \gamma$  being the digamma function and  $\gamma = 0.5772$ . The nearest Gaussian state for  $|m\rangle$ ,  $\rho_{|m\rangle}^G$ , is a thermal state with  $H_w(\rho_{|m\rangle}^G) = 1 + \log[m+1]$  [20] leading to

$$\begin{aligned} NC(|m\rangle) &= m + \log m! - m\Psi(m+1) \\ NG(|m\rangle) &= \log[m+1] - m - \log m! + m\Psi(m+1) \end{aligned} \quad (9)$$

We plot the  $\Delta(\rho_{|m\rangle}^G)$ ,  $NC(|m\rangle)$  and  $NG(|m\rangle)$  for different values of  $m$  in Fig. 1(a). For small  $m (\leq 4)$  we observe a rapid increase in both  $\Delta(\rho_{|m\rangle}^G)$  and  $NC$  while  $NG$  increases slowly. With further increase in  $m$ , both  $\Delta(\rho_{|m\rangle}^G)$ ,  $NC(|m\rangle)$  and  $NG(|m\rangle)$  increase monotonically and saturate for very high  $m$ . As  $m$  increases,  $NC(|m\rangle)$  captures the growing singularity in corresponding Glauber-Sudarshan P distribution [1] as well as the negativity in the Wigner distribution [10]. We also note that the increase in  $NC(|m\rangle)$  is almost twice of that in  $NG(|m\rangle)$ .

Next, we consider squeezed coherent state  $|\psi\rangle_{\text{SC}} = S(\zeta)|\alpha\rangle$  that has been generated under the nonclassical operation, namely quadrature squeezing  $S(\zeta) = \exp(\frac{\zeta a^{\dagger 2} - \zeta^* a^2}{2})$ , applied on a coherent state  $|\alpha\rangle$ , where  $\zeta = re^{i\theta}$ ;  $r$  and  $\theta$  being the squeezing strength and the squeezing angle respectively. Since  $|\psi\rangle_{\text{SC}}$  possesses an explicit Gaussian Wigner distribution,  $NG(|\psi\rangle_{\text{SC}})$  becomes

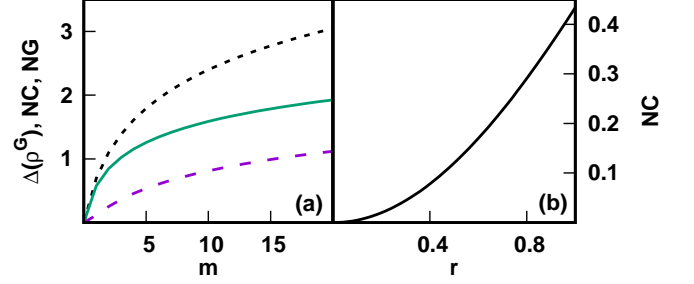


FIG. 1: (Color Online) Plot of **a**:  $\Delta(\rho_{|m\rangle}^G)$  (black, dotted line),  $NC(|m\rangle)$  (green, solid line) and  $NG(|m\rangle)$  (violet, dashed line) with  $m$ , **b**:  $NC(|\psi\rangle_{\text{SC}})$  with  $r$ .

zero. For  $|\psi\rangle_{\text{SC}}$  we obtain analytically  $\tau = \nu_r^2 + \frac{1}{2}$  and  $2\mu = \mu_r \nu_r e^{-i\theta}$ , where  $\mu_r = \cosh r$  and  $\nu_r = \sinh r$ , which yield a logarithmic  $NC$  for  $|\psi\rangle_{\text{SC}}$  as,

$$NC(|\psi\rangle_{\text{SC}}) = \log \mu_r \quad (10)$$

Evidently, the  $NC(|\psi\rangle_{\text{SC}})$  is dependent only on the  $r$  where  $\theta$  plays no active role in  $NC$  since it only sets the direction of squeezing rather than the degree of squeezing. Moreover, the coherent displacement  $\alpha$  also appears as an artifact emphasizing the fact that entropy is independent of displacement, e.g.,  $NC(S(\zeta)|\alpha\rangle) = NC(S(\zeta)|0\rangle)$ . Fig. 1(b) shows the dependence of  $NC(|\psi\rangle_{\text{SC}})$  upon  $r$ . We observe an initial slow and then rapid increase in  $NC(|\psi\rangle_{\text{SC}})$  with increase in  $r$ . However, for very high  $r$  it saturates asymptotically (not shown in the figure).

As further examples of nonclassical as well as non-Gaussian pure state, we study the even ( $|\psi_+\rangle$ ) and the odd ( $|\psi_-\rangle$ ) superposition of coherent states  $|\psi_{\pm}\rangle = \frac{1}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}(|\alpha\rangle \pm |-\alpha\rangle)$ . For the sake of simplicity we consider real displacement, e.g.,  $\alpha = R$ . We compute  $H_w(|\psi_{\pm}\rangle)$  numerically while  $H_w(\rho_{|\psi_{\pm}\rangle}^G)$  have the analytic form as,  $H_w(\rho_{|\psi_+\rangle}^G) = 1 + \frac{1}{2} \log[1 + 2R^2 \tanh R^2 - \frac{R^4}{\cosh^2 R^2}]$  and  $H_w(\rho_{|\psi_-\rangle}^G) = 1 + \frac{1}{2} \log[1 + 2R^2 \coth R^2 + \frac{R^4}{\sinh^2 R^2}]$ .

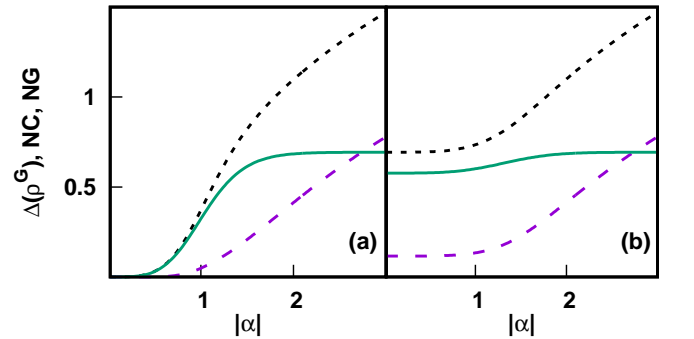


FIG. 2: (Color Online) Plot of  $\Delta(\rho^G)$  (black, dotted line),  $NC$  (green, solid line) and  $NG$  (violet, dashed line) with  $m$  for **a**:  $|\psi_+\rangle$  and **b**:  $|\psi_-\rangle$ . In high  $R$  domain ( $R \geq 2.0$ ), curves for  $\Delta(\rho^G)$  and  $NC$  become almost parallel to each other for both  $|\psi_{\pm}\rangle$ . Nonetheless, for  $|\psi_{\pm}\rangle$ , curve of  $NG$  (dashed line) crosses that for  $NC$  (solid line) at  $R \sim 2.75$ .

We show the dependence of  $\Delta(\rho^G)$ ,  $NC$  and  $NG$  on  $R$  for  $|\psi_{\pm}\rangle$  in Fig. 2. With increase in  $R$ , for  $|\psi_{+}\rangle$  [Fig. 2(a)], we observe a sharp increase in both  $\Delta(\rho^G)$  and  $NC$  while  $NG$  increases slowly upto a critical coherent amplitude  $R(\leq 1.0)$ . In contrast, for  $|\psi_{-}\rangle$  [Fig. 2(b)], we notice initial slow increase in  $\Delta(\rho^G)$ ,  $NC$  and  $NG$  upto  $R \sim 1.0$ . It is noteworthy that for small  $R(\leq 1.0)$ ,  $|\psi_{-}\rangle$  is more nonclassical than  $|\psi_{+}\rangle$ ; however, for large  $R(\sim 1.5)$  both  $|\psi_{\pm}\rangle$  contains equal amount of  $NC$ . With further increase in  $R$ , for both  $|\psi_{\pm}\rangle$ ,  $NG$  keeps increasing monotonically while  $NC$  saturates as shown in [12]. This, we expect at high  $R$  due to the increase in the distance between the coherent amplitudes with opposite phase that effectively leads to the same coherent mixed state superposition ( $\lim_{r \rightarrow \rho_{\pm}} \rho_{\pm} \rightarrow \frac{1}{2}(|R\rangle\langle R| + |-R\rangle\langle -R|)$ ) with negligible quantum correlation [12]; however, the concerned increase in the relative distance makes the corresponding  $W(z, z^*)$  more non-Gaussian. Interestingly, for small  $R$ , both  $|\psi_{\pm}\rangle$  have  $NC$  more than  $NG$  while with increase in  $R$ , their  $NG$  becomes greater than the respective  $NC$ . Here, we represent examples of pure states, in particular  $|\psi_{\pm}\rangle$ , for which  $NC$  could be greater as well as less than  $NG$ , as in Eq. (8).

*Mixed States :* We now test the validity our formalism for nonclassical mixed states, namely, photon excited and quadrature squeezed thermal state  $\rho_{th}(\bar{n})$ .

For an  $m$ -photon added thermal state ( $m$ -PATS)  $\rho_{m-PATS} = \frac{1}{(1+\bar{n})^m} a^{\dagger m} \rho_{th}(\bar{n}) a^m$ , corresponding Gaussian state  $\rho_{m-PATS}^G$  is a thermal state which results in  $NC(\rho_{m-PATS}^G) = 0$ . Thus, the simultaneous quantization of  $NC$  and  $NG$  (Eq. 8), for mixed states, yields  $NG$  of  $\rho_{m-PATS}$  as the measure of its  $NC$ . Furthermore, as  $Q_{m-PATS}(\alpha, \alpha^*) = \lambda^2 Q_{|m\rangle}(\lambda\alpha, \lambda\alpha^*)$ ;  $\lambda = \frac{1}{1+\bar{n}}$ , we have  $NG(\rho_{m-PATS}) = NG(|m\rangle)$  [20] which further yields  $NC(\rho_{m-PATS}) = NG(|m\rangle)$  Eq. (9). We plot  $NC(\rho_{m-PATS})$  for different  $m$  in Fig 3(a). With increase in  $m$ ,  $NC(\rho_{m-PATS})$  increases monotonically and saturates at very high  $m$ . It is noteworthy that,  $NC(\rho_{m-PATS})$  is completely independent of thermal fluctuation (measured in terms of  $\bar{n}$ ).

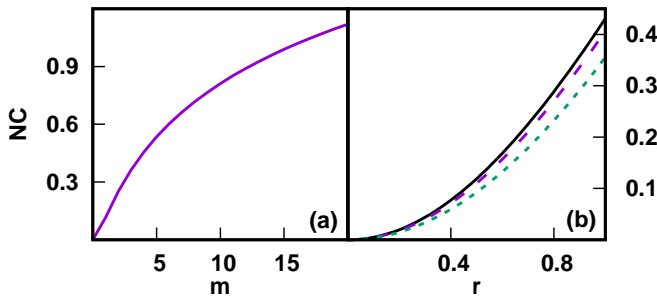


FIG. 3: (Color Online) Plot of **a**:  $NC(\rho_{m-PATS})$  with  $m$  and **b**:  $NC(\rho_{ST})$  with  $r$  for  $x = 0.1$  (black, solid line),  $0.3$  (violet, dashed line) and  $0.5$  (green, dotted line) .

On the other hand, for a squeezed thermal state  $\rho_{ST} = S(r)\rho_{th}(\bar{n})S^{\dagger}(r)$  which is a Gaussian state we have

$NG(\rho_{ST}) = 0$ . We analytically obtain the thermal state  $\rho_{th}(\bar{n})$  that is nearest to  $\rho_{ST}$  same as the initial  $\rho_{th}(\bar{n})$  upon which  $S(r)$  is applied in  $\rho_{ST}$ . Our entropic formulation of  $NC$ , for  $\rho_{ST}$ , recovers, as the classical reference state, the initial thermal state as considered in [9] leading to a logarithmic  $NC$  of the form

$$NC(\rho_{ST}) = \log[\mu_r + \bar{n}] - \log[1 + \bar{n}] \quad (11)$$

In the limit  $\bar{n} \rightarrow 0$ ,  $NC(\rho_{ST}) \rightarrow NC(S(r)|0\rangle) = \log \mu_r$  (Eq. 10) as well as while  $r \rightarrow 0$ ,  $NC(\rho_{ST}) \rightarrow NC(\rho_{th}(\bar{n})) = 0$ . We plot  $NC(\rho_{ST})$  with  $r$  for different  $x(= \frac{\bar{n}}{1+\bar{n}})$  in Fig 3(b). Evidently,  $NC(\rho_{ST})$  increases with increase in  $r$ , however on increasing  $x$  (i.e.  $\bar{n}$ ),  $NC(\rho_{ST})$  falls off. We detect the effect of squeezing for all values of  $r$  in contrast to [13] that reads  $\rho_{ST}$  nonclassical only if  $r \geq r_c(= \log[1 + 2\bar{n}])$ .

*Discussion :* We show that for pure states, the entropic definition of  $NC$  is obtained without any optimization which further yields the unified quantification of  $NC$  and  $NG$  (Eq. 6) very naturally. On the other hand, for mixed states, the choice of reference thermal state has been made phenomenologically. This could, otherwise, be done using brute force method, i.e., by minimizing the quantity  $|H_w(\rho_{nc}) - H_w(\rho_{th})|$ , where optimization runs over the set of all thermal states. However to the best of the knowledge of author, it is currently unknown whether an analytic proof can be provided due the anti-normal ordering of operators involved in Q function description. Besides our main focus is to construct the nearest thermal state from physical intuition bypassing the optimization. Although, the current work describes  $NC$  for both pure and mixed states efficiently, it remains an open question whether one can mathematically retrieve the nearest thermal state as described here. Moreover, the unified description of  $NC$  and  $NG$  for mixed states leads to an operational interpretation of  $NC$ . It can be viewed as the experimentally accessible  $NC$  for the mixed state  $\rho_{nc}$ .

For pure states, we observe that photon excitation and quadrature squeezing give rise to monotonic  $NC$  and  $NG$ , whereas Schrodinger kittens (e.g., small coherent amplitudes) formed with odd superposition are more nonclassical than that formed with even superposition; however the macroscopic cat states (large amplitude) are nonclassically equivalent, irrespective of the parity of superposition. Besides, for mixed states, while in case of photon excitation,  $NC$  depends solely on the number of photon excitation, for quadrature squeezed states,  $NC$  still depends on the thermal noise. We provide example of pure nonclassical non-Gaussian states, in particular, for the coherent superposition states, that have  $NC$  more as well as less than its  $NG$  depending upon the value of coherent amplitude. Albeit predicted in Eq. (8), for the non-Gaussian nonclassical mixed state, we find the nearest Gaussian state to be classical. We leave the question open whether there exists any nonclassical non-Gaussian mixed state for which the nearest Gaussian state is more nonclassical than the state itself.



While the distance-based definition of  $NC$  [9], requires optimization over the set of all classical states and  $NC$  defined in terms of nonclassical depth [11] fails for all non-Gaussian states [28], Wigner negativity [12] is not applicable for Gaussian states. Results also show that the description of single mode  $NC$  in terms of BS output entanglement depends upon the specific choice of entanglement potential [29] as well as BS converts the  $NC$  of Gaussian states into entanglement partially [30] and hence call in question the effectiveness of entanglement-based quantification of  $NC$  [13]. The operational approach to quantify the  $NC$ , proposed recently by Gehrke *et. al.* [31], also finds the photon number state maximally nonclassical similar to [10] as well as a squeezed state attaining maximal  $NC$  at a moderate squeezing strength. Besides, while the distance based  $NG$  [18] fails to satisfy the shape criterion [20],  $NG$  defined in terms of relative entropy [19] fails for all pure states. In contrast with using phase-space singularity and negativity [11, 12, 32, 33], we propose a unique and unified description of  $NC$  and  $NG$  in terms of the classical like distribution that can be easily computed without any optimization as well as be verified experimentally in optical heterodyne detection [34].

**Conclusion :** In summary, we show that the entropic definition of  $NC$ , explains the conjugation of non-Gaussian and nonclassical character of both pure and mixed states of quantized light under the action of different  $NC$ -inducing operations, *viz.*, photon excitation, quadrature squeezing as well as quantum superposition. While it recovers the earlier results qualitatively, requiring no optimization, in the same framework, current formulation explores new possibilities in the context of simultaneity of  $NC$  and  $NG$ .

Current formalism, by using the description of Q-function in finite-dimension [35], can be extended to the finite dimensional quantum systems [36] alongwith macroscopic optomechanical systems [37] and thus, sets the framework for studying the conversion of  $NC$  into entanglement by the action of BS, in general context [38]. Furthermore, due to the use of classical like distribution, our work becomes very crucial in analyzing the evolution of  $NC$  and  $NG$  of optical states under quantum processes [39, 40]. Consequently, present formulation becomes important in understanding the quantum-classical transition in open quantum systems [41] alongwith the role of  $NC$  and  $NG$  of quantum states in several information tasks processing with BS generated entanglement such as entanglement distillation [42], entanglement distribution [43, 44], quantum computation [45, 46] *etc* as well as systems of biological interest [47].

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- [1] W. P. Schleich, *Quantum Optics in Phase Space*, (1<sup>st</sup> Edition, Wiley-VCH, Berlin, 2001)
- [2] R. J. Galuber, Phys. Rev. **131**, 2766 (1963)
- [3] G. S. Agarwal and K. Tara, Phys. Rev. A **43**, 492 (1991)
- [4] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976)
- [5] Y. Yamamoto, N. Imoto and S. Machida, Phys. Rev. A **33**, 3243 (1986)
- [6] M. Hillery, Phys. Lett. A **111**, 409 (1985)
- [7] G. S. Agarwal and E. Wolf, Phys. Rev. D **2**, 2161(1970)
- [8] G. S. Agarwal and G. Adam, Phys. Rev. A **39**, 6259 (1989)
- [9] M. Hillery, Phys. Rev. A **35**, 725 (1987)
- [10] V. V. Dodonov, O. V. Manko, V. I. Manko and A. Wunsche, J.Mod.Opt. **47**, 633 (2000)
- [11] C. T. Lee, Phys. Rev. A(R) **44**, R2775 (1991)
- [12] A. Kenfack and K. Życzkowski, J. Opt. B:Quantum Semi-class. Opt. **6**, 396 (2004)
- [13] J. K. Asboth, J. Calsamiglia and H. Ritsch, Phys. Rev. Lett. **94**, 173602 (2005)
- [14] G. S. Agarwal and K. Tara, Phys. Rev. A **46**, 485 (1992)
- [15] D. N. Klyshko, Phys. Lett. A **213**, 7 (1996)
- [16] Arvind, N. Mukunda and R. Simon, Phys. Rev. A **56**, 5042 (1997)
- [17] T. Richter and W. Vogel, Phys. Rev. Lett. **89**, 283601 (2002)
- [18] M. G. Genoni, M. G. A. Paris and K. Banaszek, Phys. Rev. A **76**, 042327 (2007)
- [19] M. G. Genoni, M. G. A. Paris and K. Banaszek, Phys. Rev. A **78**, 060303 (R) (2008)
- [20] J. S. Ivan, M. S., Kumar and R. Simon, Quant. Inf. Process **11**, 853 (2012)
- [21] M. G. Genoni, M. L. Palma, T. Tufarelli, S. Olivares, M. S. Kim and M. G. A. Paris, Phys. Rev. A **87**, 062104 (2013)
- [22] C. Hughes, M. G. Genoni, T. Tufarelli, M. G. A. Paris and M. S. Kim, Phys. Rev. A **90**, 013810 (2014)
- [23] J. Park *et al.*, Phys. Rev. Lett. **114**, 190402 (2015)
- [24] By definition, Husimi  $Q(\beta, \beta^*)$  distribution is well defined, positive semi-definite ( $Q(\beta, \beta^*) \geq 0$ ) and satisfies all the properties of a classical probability distribution.
- [25] A. Ourjoumtsev, R. Tualle-Broui, J. Laurat and P. Grangier, Science **312**, 83 (2006)
- [26] E. H. Lieb, Commun. Math. Phys. **62**, 35 (1978)
- [27] P. Marian and T. A. Marian, Phys. Rev. A **88**, 012322 (2013)
- [28] N. Lutkenhaus and S. M. Barnett, Phys. Rev. A **51**, 3340 (1995)
- [29] A. Miranowicz, K. Bartkiewicz, N. Lambert, Yueh-Nan Chen and F. Nori, Phys. Rev. A **92**, 062314 (2015)
- [30] W. Ge, M. E. Tasgin and M. S. Zubairy, Phys. Rev. A **92**, 052328 (2015)
- [31] C. Gehrke, J. Sperling and W. Vogel, Phys. Rev. A **86**, 052118 (2012)
- [32] T. Kiesel, Phys. Rev. A **87**, 062114 (2013)
- [33] E. Agudelo, J. Sperling, W. Vogel, S. Kohnke, M. Mraz and B. Hage, Phys. Rev. A **92**, 033837 (2015)
- [34] Z. Y. Ou and H. J. Kimble, Phys. Rev. A **52**, 3126 (1995)
- [35] T. Opatrny, V. Buzek, J. Bajer and G. Drobný, Phys. Rev. A **52**, 2419 (1995)
- [36] F. Bohnet-Waldraff, D. Braun and O. Giraud, Phys. Rev. A **93**, 012104 (2016)
- [37] F. Khalili, S. Danilishin, H. Miao, H. Muller-Ebhardt, H. Yang and Y. Chen, Phys. Rev. Lett. **105**, 070403 (2010)
- [38] N. Killoran, F. E. S. Steinhoff and M. B. Plenio, Phys. Rev. Lett. **116**, 080402 (2016)
- [39] S. Rahimi-Keshari, T. Kiesel, W. Vogel, S. Grandi, A.

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- Zavatta and M. Bellini, Phys. Rev. Lett. **110**, 160401 (2013)
- [40] K. K. Sabapathy, Phys. Rev. A **93**, 042103 (2016)
- [41] Li-yun Hu, Xue-xiang Xu, Zi-sheng Wang and Xue-fen Xu, Phys. Rev. A **82**, 043842 (2010)
- [42] J. S. Ivan, N. Mukunda and R. Simon, Quantum Inf. Process. **11**, 873 (2012)
- [43] Z. Jiang, M. D. Lang and C. M. Caves, Phys. Rev. A **88**, 044301 (2013)
- [44] C. Croal *et al.*, Phys. Rev. Lett. **115**, 190501 (2015)
- [45] V. Veitch, N. Wiebe, C. Ferrie and J. Emerson, New. J. Phys. **15**, 013037 (2013)
- [46] H. Pashayan, J. J. Wallman and S. D. Barlett, Phys. Rev. Lett. **115**, 070501 (2015)
- [47] E. J. O'Reilly and A. Olaya-Castro, DOI: 10.1038/ncomms4012 (2014)